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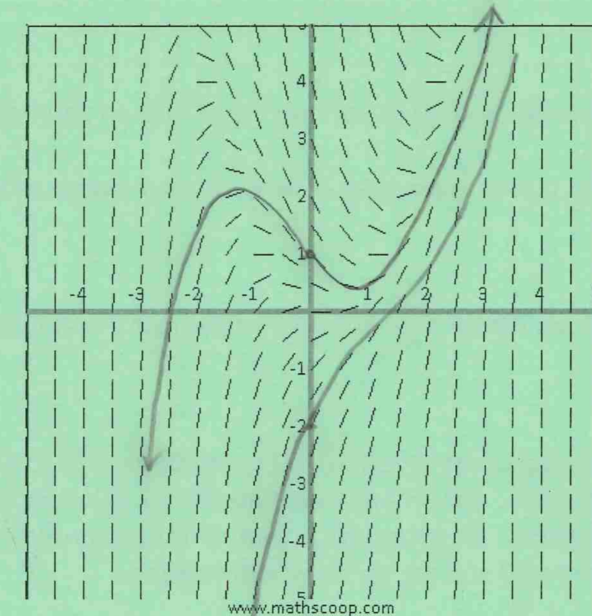
Differential Equations Test 1 Name Key

Work each of the following problems analytically on the blank paper provided. Show all work in a neat, orderly manner. No credit will be given for answers without the work involved in arriving at the answer.

1. On the graph shown, sketch an appropriate solution curve that passes through each of the indicated points. Label each sketch with the letter of the point.

   4 A) $y(0) = 1$

   4 B) $y(0) = -2$



4 2. A) Show that $y = \ln(x) + c$ is a solution to the DE $xy'' + y' = 0$ on the interval $(0, \infty)$.

   4 B) Given the general solution in part A, find the solution to the initial-value problem consisting of the given DE and the initial condition $y(2) = 4$.

6 3. Find c_1 and c_2 so that $y(x) = c_1 \sin(x) + c_2 \cos(x)$ will satisfy the conditions $y(0) = 1$ and

   $y'(\pi) = 4$.

4. Consider the DE: $\dfrac{dy}{dx} = y(y - 6)(y + 1)^2$

   7 A) Construct a phase portrait for the differential equation.

   3 B) Classify the critical points as stable, semi-stable, or unstable.

5. Use separation of variables to solve the initial value problem:

10 $$\frac{dy}{dx} = \frac{e^{3x}+x}{y^2}; \; y(0) = 5$$

6. Show that the DE is exact, and use the method of exact DE's to solve.

10 $$(\cos(y)+e^x)dx - (x\sin(y))dy = 0$$

52

(4)

2\. A) $y = \ln(x) + c$

$y' = \frac{1}{x}$

$y'' = -\frac{1}{x^2}$

$xy'' + y' = 0$

$x\left(-\frac{1}{x^2}\right) + \frac{1}{x} = 0$

$-\frac{1}{x} + \frac{1}{x} = 0$ ✓

(4)

B) $y = \ln x + c$

$4 = \ln(2) + c$

$c = 4 - \ln(2)$

$y = \ln x + 4 - \ln(2)$

$y = \ln(x) + 3.31$

(6) 3. $y = c_1 \sin(x) + c_2 \cos(x)$

$y(0) = c_1 \sin(0) + c_2 \cos(0) = 1$

$\Rightarrow \boxed{c_2 = 1}$

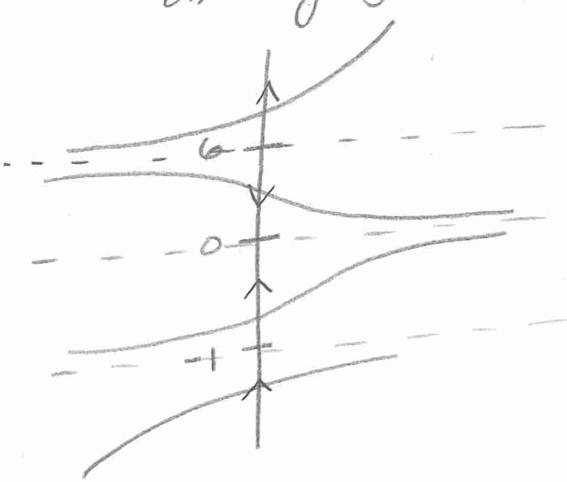
$y' = c_1 \cos(x) - c_2 \sin(x)$

$y'(\pi) = 4 \Rightarrow c_1 \cos(\pi) - c_2 \sin(\pi) = 4$

$-c_1 = 4 \Rightarrow \boxed{c_1 = -4}$

(7)

4A) $\frac{dy}{dx} = y(y-6)(y+1)^2$



B) (3)

$y = 6$ unstable

$y = 0$ stable

$y = -1$ semi-stable

5. (10)

$$\frac{dy}{dx} = \frac{e^{3x}+x}{y^2} \qquad y(0)=5$$

$$\int y^2\,dy = \int (e^{3x}+x)\,dx$$

$$\frac{1}{3}y^3 = \frac{1}{3}e^{3x} + \frac{1}{2}x^2 + C \qquad (7)$$

$$y(0)=5 \Rightarrow \quad \frac{125}{3} = \frac{1}{3} + C$$

$$C = \frac{124}{3} \qquad (3)$$

$$\frac{1}{3}y^3 = \frac{1}{3}e^{3x} + \frac{1}{2}x^2 + \frac{124}{3}$$

$$y^3 = e^{3x} + \frac{3}{2}x^2 + 124$$

$$y = \sqrt[3]{e^{3x} + 3/2\,x^2 + 124}$$

6. (10)

$$\left.\begin{aligned} M_y &= -\sin(y) \\ N_x &= -\sin(y) \end{aligned}\right\} 4$$

$$\left.\begin{aligned} f_x &= \cos(y) + e^x \\ f(x,y) &= x\cos(y) + e^x + h(y) \\ f_y &= -x\sin(y) + h'(y) \\ &\Rightarrow h'(y) = 0 \qquad h(y) = c \end{aligned}\right\} (6)$$

$$x\cos(y) + e^x + c = 0$$

7. $xy\,dx + (2x^2+3y^2-1)\,dy = 0$

(8)

$M_y = x \qquad N_x = 4x$

$\frac{M_y - N_x}{N} = \frac{x-4x}{2x^2+3y^2-1}$ No

$\frac{N_x - M_y}{M} = \frac{4x-x}{xy} = \frac{3}{y}$ Yes

$I(y) = e^{\int \frac{3}{y}dy} = e^{3\ln y} = y^3$ (6)

$xy^4\,dx + (2x^2y^3 + 3y^5 - y^3)\,dy = 0$

(2) $M_y = 4xy^3 \qquad N_x = 4xy^3$ ✓

8. $\frac{dy}{dx} - \frac{1}{x^2}y = e^{-1/x}$ $\qquad I = e^{\int -\frac{1}{x^2}dx} = e^{1/x}$ (3)

(8)

$e^{1/x}\frac{dy}{dx} - \frac{1}{x^2}e^{1/x}y = 1$

$\frac{d}{dx}\left[e^{1/x}y\right] = 1$ (5)

$e^{1/x}y = x + c$

$y = (x+c)e^{-1/x}$

(10)

9. $\frac{(ty)^2 + (tx)(ty)}{(tx)^2} = \frac{y^2+xy}{x^2}$ ✓ (3)

$y = xv$

$\frac{dy}{dx} = v + \frac{dv}{dx}x$

(7) $v + x\frac{dv}{dx} = \frac{(xv)^2 + x(xv)}{x^2}$

$v + x\frac{dv}{dx} = v^2 + v$

$x\frac{dv}{dx} = v^2$

$\int \frac{1}{v^2}dv = \int \frac{1}{x}dx$

$-\frac{1}{v} = \ln|x| + c$

$-\frac{x}{y} = \ln|x| + c$

$y = \frac{-x}{\ln|x| + c}$

10A) $T(0)=180$ $\qquad \frac{dT}{dt}=k(T-75)$

(4) $T_m=75$

$T(7)=100$

(5) B) $\frac{1}{T-75}\,dT = k\,dt$

$\ln|T-75| = kt+c$

$T-75 = e^{kt+c}$

$T-75 = c_1 e^{kt}$

$$\boxed{T=105e^{\frac{\ln(5/21)}{7}t}+75}$$

$T(0)=180 \qquad 180-75=c_1$

$105=c_1$

$T-75=105e^{kt}$

$T(7)=100 \qquad 100-75=105e^{k(7)}$

$\frac{25}{105}=e^{7k}$

$k=\frac{\ln\left(\frac{5}{21}\right)}{7}$

11A) $A(0)=40$ $\qquad R_{in}=(2\ \text{lb/gal})(2\ \text{gal/min})=4\ \text{lb/min}$

(4) $\qquad R_{out}=(4\ \text{gal/min})\left(\frac{A}{100-2t}\right)=\frac{2A}{50-t}$

$\frac{dA}{dt}=4-\frac{2A}{50-t}$

$I=e^{\int \frac{2}{50-t}dt}=e^{-2\ln(50-t)}$

$=\frac{1}{(50-t)^2}$

(5) B) $\frac{dA}{dt}+\frac{2A}{50-t}=4$

$\frac{1}{(50-t)^2}\frac{dA}{dt}+\frac{2A}{(50-t)^3}=\frac{4}{(50-t)^2}$

$\frac{d}{dt}\left(\frac{1}{(50-t)^2}A\right)=\frac{4}{(50-t)^2}$

$A(0)=40 \Rightarrow \frac{1}{2500}(40)-\frac{4}{50}=c$

$c=\frac{-8}{125}$

$\approx -.064$

$\frac{1}{(50-t)^2}A=\frac{4}{50-t}+C$

$A=4(50-t)-\frac{8}{125}(50-t)^2$

12. $\frac{dA}{dt} = rA$ $\qquad A(10) = 2A_0 \qquad$ Find r.

(1) $\qquad A(0) = A_0$

$\frac{1}{A}\, dA = r\, dt$

$\ln|A) = rt + c$

$A = e^{rt+c}$

$A = A_0 e^{rt}$

$2A_0 = A_0 e^{r(10)}$

$2 = e^{10r}$

$\ln(2) = 10r$

$\frac{\ln(2)}{10} = r$

$r \approx .069$ or 6.9% interest